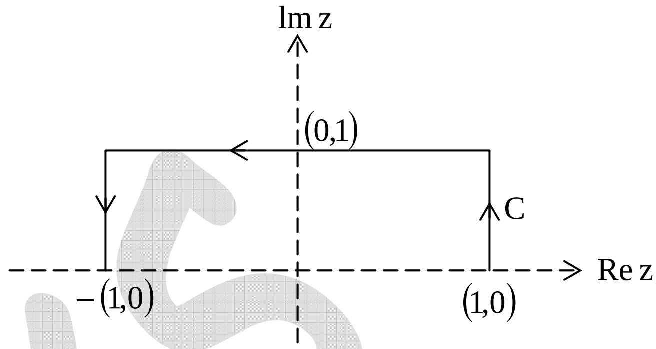
MATEMATICAL PHYSICS SOLUTIONS

NET/JRF (JUNE-2011)

Q1. The value of the integral $\int_C dz\, z^2 e^z$, where C is an open contour in the complex z -plane as shown in the figure below, is:



(a) $\frac{5}{e}+e$ (b) $e-\frac{5}{e}$

(c) $\frac{5}{e}-e$ (d) $-\frac{5}{e}-e$

Ans: (c)

Solution: If we complete the contour, then by Cauchy integral theorem

$$\int_{-1}^{1} dzz^2e^z + \int_C dzz^2e^z = 0 \Rightarrow \int_C dzz^2e^z = -\int_{-1}^{1} dzz^2e^z = -\left[z^2e^z - 2ze^z + 2e^z\right]_{-1}^{1} = \frac{5}{e} - e$$

Q2. Which of the following matrices is an element of the group $SU(2)$?

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & i \\ 3 & 1+i \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Ans: (b)

Solution: $SU(2)$ is a group defined as following: $SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in C; |\alpha|^2 + |\beta|^2 = 1 \right\}$

clearly (b) hold the property of $SU(2)$. $\alpha = \frac{1+i}{\sqrt{3}}, \beta = \frac{1}{\sqrt{3}}$ and $\overline{\alpha} = \frac{1-i}{\sqrt{3}}, \overline{\beta} = \frac{1}{\sqrt{3}}$.

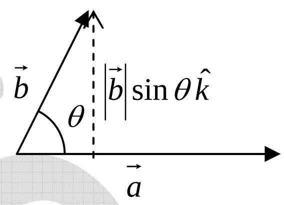
Note: $SU(2)$ has wide applications in electroweak interaction covered in standard model of particle physics.

Q3. Let $\vec{a}$ and $\vec{b}$ be two distinct three dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a}\times\left(\vec{b}\times\vec{a}\right)}{a^2}$ (b) $\frac{\vec{b}\times\left(\vec{a}\times\vec{b}\right)}{b^2}$ (c) $\frac{\left(\vec{a}\cdot\vec{b}\right)\vec{b}}{b^2}$ (d) $\frac{\left(\vec{b}\cdot\vec{a}\right)\vec{a}}{a^2}$

Ans: (a)

Solution: $\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$ where $\hat{n}$ is perpendicular to plane containing $\vec{a}$ and $\vec{b}$ and pointing upwards.



$$\vec{a}\times\left(\vec{a}\times\vec{b}\right) = ab\sin\theta\left(\vec{a}\times\hat{n}\right) = -a^2 b\sin\theta\,\hat{k}$$

$$b\sin\theta\,\hat{k} = \frac{-\vec{a}\times\left(\vec{a}\times\vec{b}\right)}{a^2} \Rightarrow b\sin\theta\,\hat{k} = \frac{\vec{a}\times\left(\vec{b}\times\vec{a}\right)}{a^2}.$$

Q4. Let $p_n(x)$ (where $n = 0,1,2,......$) be a polynomial of degree n with real coefficients, defined in the interval $2 \le n \le 4$. If $\int_2^4 p_n(x)p_m(x)dx = \delta_{nm}$, then

(a) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$ (b) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$

(c) $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$ (d) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

Ans: (d)

Solution: For n not equal to m kroneker delta become zero. One positive and one negative term can make integral zero. So answer may be (c) or (d). Now take $n = m = 0$ so $p_0(x) = \frac{1}{\sqrt{2}}$ and then integrate. (d) is correct option because it satisfies the equation Check by integration and by orthogonal property of Legendre polynomial also.

Q5. Which of the following is an analytic function of the complex variable $z = x + iy$ in the domain $|z| < 2$?

(a) $(3 + x - iy)^7$ (b) $(1 + x + iy)^4 (7 - x - iy)^3$

(c) $(1 - x - iy)^4 (7 - x + iy)^3$ (d) $(x + iy - 1)^{1/2}$

Ans: (b)

Solution: Put $z = x + iy$. If $\overline{z} = x - iy$ appears in any of the expressions then that expression is non-analytic. For option (d) we have a branch point singularity as the power is $\frac{1}{2}$ which is fractional. Hence only option (b) is analytic.

Q6. Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

A. The eigenvalues of M are

(a) 0, 1, 2 (b) 0, 0, 3 (c) 1, 1, 1 (d) – 1, 1, 3

Ans: (b)

Solution: For eigen values $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = 0$

$$(1-\lambda)\left((1-\lambda)^2 - 1\right) - (1 - \lambda - 1) + 1\left(1 - (1-\lambda)\right) = 0$$

$$(1-\lambda)(1 + \lambda^2 - 2\lambda - 1) + \lambda + \lambda = 0 \Rightarrow \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 2\lambda = 0$$

$$\lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0 \Rightarrow \lambda = 0, 0, 3$$

For any $n \times n$ matrix having all elements unity eigenvalues are $0, 0, 0, \ldots, n$.

B. The exponential of M simplifies to (I is the 3 × 3 identity matrix)

(a) $e^M = I + \left(\frac{e^3 - 1}{3}\right) M$ (b) $e^M = I + M + \frac{M^2}{2!}$

(c) $e^M = I + 3^3 M$ (d) $e^M = (e - 1) M$

Ans: (a)

Solution: For e^M let us try to diagonalize matrix M using similarity transformation.

$$\text{For } \lambda = 3, \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0,\ x_1 - 2x_2 + x_3 = 0,\ x_1 + x_2 - 2x_3 = 0$$

$$\Rightarrow -3x_2 + 3x_3 = 0 \text{ or } x_2 = x_3 \Rightarrow x_1 = x_2 = x_3 = k.$$

Eigen vector is $1\sqrt{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ where $k = 1$.

For $\lambda = 0$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$$

Let $x_1 = k_1, x_2 = k_2$ and $x_3 = k_1 + k_2$. Eigen vector is $\begin{bmatrix} k_1 \\ k_2 \\ (k_1+k_2) \end{bmatrix} = 1/\sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ where $k_1 = k_2 = 1$.

Let $x_1 = k_1, x_2 = k_2$ and $x_3 = -(k_1 + k_2)$. Other Eigen vector $1/\sqrt{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ where $k_1 = 1, k_2 = -1$.

$$S = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow D = S^{-1}MS, M = SDS^{-1}.$$

$$e^M = Se^D S^{-1} \Rightarrow e^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^3 \end{bmatrix} \Rightarrow e^M = 1 + \frac{(e^3 - 1)M}{3}$$

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Q7. An unbiased dice is thrown three times successively. The probability that the numbers of dots on the uppermost surface add up to 16 is

(a) $\frac{1}{16}$ (b) $\frac{1}{36}$ (c) $\frac{1}{108}$ (d) $\frac{1}{216}$

Ans: (b)

Solution: We can get sum of dice as 16 in total six ways i.e. three ways (6, 5, 5) and three ways (6, 6, 4).

Total number of ways for 3 dice having six faces $= 6\times 6\times 6$

$$= \frac{6}{6\times 6\times 6} = \frac{1}{36}$$

Q8. The generating function $F(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t) = \left(1-2xt+t^2\right)^{-1/2}$. The value of $P_3(-1)$ is

(a) $5/2$ (b) $3/2$ (c) $+1$ (d) -1

Ans: (d)

Solution: $P_3 = \frac{1}{2}\left(5x^3 - 3x\right) \Rightarrow P_3(-1) = \frac{1}{2}\left(5(-1)^3 - 3(-1)\right) = \frac{1}{2}[-5+3] = -1$

Q9. The equation of the plane that is tangent to the surface $xyz = 8$ at the point $(1,2,4)$ is

(a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$

(c) $x + 4y + 2 = 0$ (d) $x + y + z = 7$

Ans: (b)

Solution: To get a normal at the surface lets take the gradient

$$\vec{\nabla}(xyx) = yz\hat{i} + zx\hat{j} + \hat{k}xy = 8\hat{i} + 4\hat{j} + 2\hat{k}$$

We want a plane perpendicular to this so: $\left(\vec{r} - \vec{r_0}\right)\cdot\frac{\left(8\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{64+16+4}} = 0$.

$$\left[(x-1)\hat{i} + (y-2)\hat{j} + (z-4)\hat{k}\right]\cdot\left[8\hat{i} + 4\hat{j} + 2\hat{k}\right] = 0 \Rightarrow 4x + 2y + z = 12.$$

Q10. A 3×3 matrix M has $Tr[M]=6$, $Tr[M^2]=26$ and $Tr[M^3]=90$. Which of the following can be a possible set of eigenvalues of M ?

(a) $\{1,1,4\}$ (b) $\{-1,0,7\}$ (c) $\{-1,3,4\}$ (d) $\{2,2,2\}$

Ans: (c)

Solution: $Tr[M^2]=(-1)^2+(3)^2+(4)^2$ also $Tr[M^3]=(-1)^3+(3)^3+(4)^3=90$.

Q11. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+f(t)x=0$ and let $w(t)=x_1(t)\frac{dx_2(t)}{dt}-x_2(t)\frac{dx_1(t)}{dt}$. If $w(0)=1$, then $w(1)$ is given by

(a) 1 (b) e^2 (c) $1/e$ (d) $1/e^2$

Ans: (d)

Solution: $W(t)$ is Wronskian of D.E.

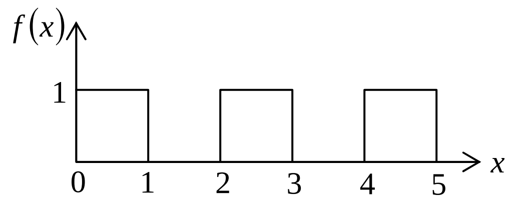
$$W=e^{-\int Pdt}=e^{-2t}\Rightarrow W(1)=e^{-2} \text{ since } P=2.$$

Q12. The graph of the function $f(x)=\begin{cases}1 & \text{for } 2n\le x\le 2n+1\\ 0 & \text{for } 2n+1\le x\le 2n+2\end{cases}$

where $n=(0,1,2,......)$ is shown below. Its Laplace transform $\tilde{f}(s)$ is

(a) $\frac{1+e^{-s}}{s}$ (b) $\frac{1-e^{-s}}{s}$

(c) $\frac{1}{s(1+e^{-s})}$ (d) $\frac{1}{s(1-e^{-s})}$



Ans: (c)

Solution: $L(f(x))=\int_0^\infty e^{-sx}f(x)dx=\int_0^1 e^{-sx}\cdot 1dx+\int_1^2 e^{-sx}\cdot 0dx+\int_2^3 e^{-sx}\cdot 1dx+......$

$$=\left[\frac{e^{-sx}}{-s}\right]_0^1+0+\left[\frac{e^{-sx}}{-s}\right]_2^3+......=\frac{1}{-s}\left[e^{-s}-1\right]+\frac{1}{-s}\left[e^{-3s}-e^{-2s}\right]+......$$

$$=\frac{1}{-s}\left[-1+e^{-s}-e^{-2s}+e^{-3s}+........\right]=\frac{1}{s}\left[1-e^{-s}+e^{-2s}-e^{-3s}+....\right]$$

Since $S_\infty=\frac{a}{1-r}$ where $r=-e^{-s}$ and $a=1 \Rightarrow S_\infty=\frac{1}{s}\left[\frac{1}{\left(1+e^{-s}\right)}\right]$.

Q13. The first few terms in the Taylor series expansion of the function $f(x)=\sin x$ around $x=\frac{\pi}{4}$ are:

(a) $\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)+\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3.....\right]$

(b) $\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3.....\right]$

(c) $\left[\left(x-\frac{\pi}{4}\right)-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3.....\right]$

(d) $\frac{1}{\sqrt{2}}\left[1-x+\frac{x^2}{2!}-\frac{x^3}{3!}.....\right]$

Ans: (c)

Solution: $f(x)=\sin x$

$$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

So Taylor's series is given by

$$\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3.....\right]$$

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Q14. A vector perpendicular to any vector that lies on the plane defined by $x+y+z=5$, is

(a) $\hat{i}+\hat{j}$ (b) $\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}+\hat{k}$ (d) $2\hat{i}+3\hat{j}+5\hat{k}$

Ans: (c)

Solution: Let $\phi = x+y+z-5 \Rightarrow \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z-5) = \hat{i}+\hat{j}+\hat{k}$.

Q15. The eigen values of the matrix $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}$ are

(a) $(1,4,9)$ (b) $(0,7,7)$ (c) $(0,1,13)$ (d) $(0,0,14)$

Ans: (d)

Solution: For eigenvalues $|A-\lambda I| = 0 \Rightarrow \begin{bmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 6\\ 3 & 6 & 9-\lambda\end{bmatrix} = 0$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]-2[2(9-\lambda)-18]+3[12-3(4-\lambda)]=0$$

$$(1-\lambda)(4-\lambda)(9-\lambda)-36(1-\lambda)-4(9-\lambda)+36+9\lambda=0$$

$$\lambda^3-14\lambda^2=0 \Rightarrow \lambda^2(\lambda-14)=0 \Rightarrow \lambda = 0,\ 0,\ 14 .$$

Q16. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1\le |z| \le 2$ and around $z=1$ is

(a) $\frac{1}{2}\left[1+z+z^2+....\right]\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+....\right]$ (b) $\frac{1}{1-z}+z-(1-z)^2+(1-z)^3+....$

(c) $\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right]\left[1+\frac{2}{z}+\frac{4}{z^2}+....\right]$ (d) $2(z-1)+5(z-1)^2+7(z-1)^3+....$

Ans:

Solution: $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{1-z}+\frac{1}{(z-1)-1}=\frac{1}{1-z}-(1+(1-z))^{-1}$

$$=\frac{1}{1-z}-\left[1-(1-z)+\frac{(-1)(-2)}{21}(1-z)^2+\frac{(-1)(-2)(-3)}{31}(1-z)^3\ldots\right]$$

$$=\frac{1}{1-z}-\left[z+(1-z)^2-(1-z)^3+\ldots.\right]$$

Q17. Let $u(x,y)=x+\frac{1}{2}(x^2-y^2)$ be the real part of analytic function $f(z)$ of the complex variable $z=x+iy$. The imaginary part of $f(z)$ is

(a) $y+xy$ (b) xy (c) y (d) y^2-x^2

Ans: (a)

Solution: $u(x,y)=x+\frac{1}{2}(x^2-y^2)$, $v(x,y)=?$

Check $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial y}=1+x$ $\quad v=y+xy+f(x)$

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow \frac{\partial v}{\partial x}=+y$ $\quad v=yx+f(y)$

$y+xy+f(x)=yx+f(y)$

$f(x)=0$ $\quad f(y)=y$

$V=xy+y$

Q18. Let $y(x)$ be a continuous real function in the range 0 and 2π, satisfying the inhomogeneous differential equation: $\sin x\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}=\delta\left(x-\frac{\pi}{2}\right)$

The value of dy/dx at the point $x=\pi/2$

(a) is continuous (b) has a discontinuity of 3

(c) has a discontinuity of 1/3 (d) has a discontinuity of 1

Ans: (d)

Solution: After dividing by $\sin x$, $\dfrac{d^2y}{dx^2}+\cot x\dfrac{dy}{dx}=\operatorname{cosec} 2\delta\left(x-\dfrac{x}{2}\right)$

Integrating both sides, $\dfrac{dy}{dx}+\int \cot x\left(\dfrac{dy}{dx}\right)dx=\int \operatorname{cosec} x\delta\left(x-\dfrac{\pi}{2}\right)dx$

$\dfrac{dy}{dx}+\cot x\cdot y-\int \operatorname{cosec} x\cdot ydx=1$

Using Dirac delta property: $\int f(x)\delta(x-x_0)=f(x_0)$ (it lies with the limit).

$\dfrac{dy}{dx}+y\cdot\dfrac{\cos x}{\sin x}+\int y\sin^2 xdx=1$, at $x=\pi;\sin x=0$. So this is point of discontinuity.

Q19. A ball is picked at random from one of two boxes that contain 2 black and 3 white and 3 black and 4white balls respectively. What is the probability that it is white?

(a) $34/70$ (b) $41/70$ (c) $36/70$ (d) $29/70$

Ans: (b)

Solution: Probability of picking white ball

$\boxed{2B\quad 3W}$ I $\boxed{3B\quad 4W}$ II

From box $I=\dfrac{3}{5}$ and from box $II=\dfrac{4}{7}$

Probability of picking a white ball from either of the two boxes is $=\dfrac{1}{2}\left[\dfrac{3}{5}+\dfrac{4}{7}\right]=\dfrac{41}{70}$

Q20. The eigenvalues of the antisymmetric matrix,

$$A=\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

where n_1, n_2 and n_3 are the components of a unit vector, are

(a) $0,i,-i$ (b) $0,1,-1$

(c) $0,1+i,-1,-i$ (d) $0,0,0$

Ans: (a)

Solution: $A=\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \Rightarrow -A^T=\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$\Rightarrow \lambda_1=0 \qquad \Rightarrow \lambda_2=-\sqrt{-n_1^2-n_2^2-n_3^2} \qquad \Rightarrow \lambda_3=\sqrt{-n_1^2-n_2^2-n_3^2}$

but $\sqrt{n_1^2+n_2^2+n_3^2}=1$

so, $\quad \lambda_1=0,\ \lambda_2=L,\ \lambda_3=-L$

$A=-A^T$ (Antisymmetric). Eigenvalues are either zero or purely imaginary.

Q21. Which of the following limits exists?

(a) $\lim\limits_{N\to\infty}\left(\sum\limits_{m=1}^{N}\frac{1}{m}+\ln N\right)$ (b) $\lim\limits_{N\to\infty}\left(\sum\limits_{m=1}^{N}\frac{1}{m}-\ln N\right)$

(c) $\lim\limits_{N\to\infty}\left(\sum\limits_{m=1}^{N}\frac{1}{\sqrt{m}}-\ln N\right)$ (d) $\lim\limits_{N\to\infty}\sum\limits_{m=1}^{N}\frac{1}{m}$

Ans: (b)

Q22. A bag contains many balls, each with a number painted on it. There are exactly n balls which have the number n (namely one ball with 1, two balls with 2, and so on until N on them). An experiment consists of choosing a ball at random, noting the number on it and returning it to the bag. If the experiment is repeated a large number of times, the average value the number will tend to

(a) $\frac{2N+1}{3}$ (b) $\frac{N}{2}$ (c) $\frac{N+1}{2}$ (d) $\frac{N(N+1)}{2}$

Ans: (a)

Solution: Total number of balls $1+2+3+4+.....+N=\frac{N(N+1)}{2}$

The probability for choosing a k^{th} ball at random $=\frac{k}{\frac{N(N+1)}{2}}$

Average of it is given by $\langle k\rangle = \Sigma k \cdot P = \dfrac{2\Sigma k^2}{N(N+1)} = \dfrac{2}{N(N+1)} \cdot \dfrac{N(N+1)(2N+1)}{6}$

$$= \frac{2N+1}{3} \quad \text{where } \Sigma k^2 = \frac{N(N+1)(2N+1)}{6}.$$

Q23. Consider a sinusoidal waveform of amplitude $1V$ and frequency f_0. Starting from an arbitrary initial time, the waveform is sampled at intervals of $\dfrac{1}{2f_0}$. If the corresponding Fourier spectrum peaks at a frequency $\bar{f}$ and an amplitude $\bar{A}$, them

(a) $\bar{f} = 2f_0$ and $\bar{A} = 1V$

(b) $\bar{f} = 2f_0$ and $0 \le \bar{A} \le 1\bar{A}$

(c) $\bar{f} = 0$ and $\bar{A} = 1V$

(d) $\bar{f} = \dfrac{f_0}{2}$ and $\bar{A} = \dfrac{1}{\sqrt{2}}V$

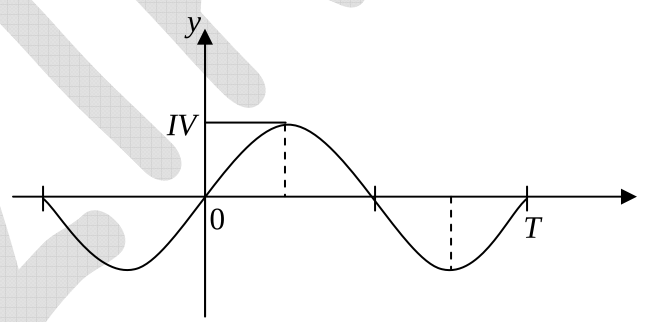
Ans: (b)

Solution: $y = 1\sin(2\pi f_0 t)$.

The fourier transform is:

$$F(y) = \frac{1}{2}[\delta(f+f_0)] - \delta[f-f_0]$$

In Fourier domain $\bar{f} = f_0, \bar{A} = \dfrac{1}{2}$.



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Q24. The unit normal vector of the point $\left[\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is

(a) $\dfrac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(b) $\dfrac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(c) $\dfrac{b\hat{i} + c\hat{j} + a\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(d) $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Ans: All the options given are incorrect.

Solution: Here $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$.

Unit normal vector is $\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$.

So, $\vec{\nabla}\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = \frac{2x\hat{i}}{a^2} + \frac{2y\hat{j}}{b^2} + \frac{2z\hat{k}}{c^2}$

$$\vec{\nabla}\phi\Big|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}$$

$$\left|\vec{\nabla}\phi\right| = \sqrt{\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{4}{3c^2}} = \frac{2}{\sqrt{3}}\sqrt{\frac{b^2c^2 + a^2c^2 + a^2c^2}{a^2b^2c^2}}$$

$$\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}\Bigg|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}}{\frac{2}{\sqrt{3}}\frac{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}{abc}} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

Q25. Given a 2×2 unitary matrix U satisfying $U^\dagger U = UU^\dagger = 1$ with $\det U = e^{i\varphi}$, one can construct a unitary matrix $V\left(V^\dagger V = VV^\dagger = 1\right)$ with $\det V = 1$ from it by

(a) multiplying U by $e^{-i\varphi/2}$

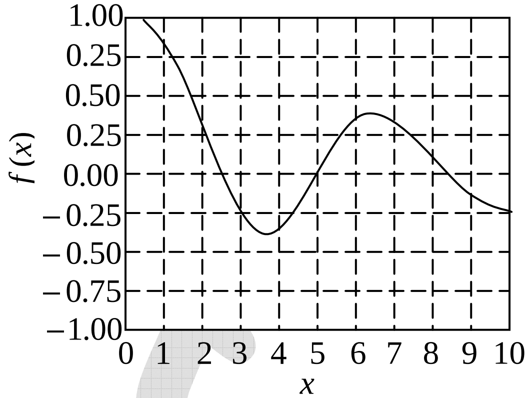
(b) multiplying any single element of U by $e^{-i\varphi}$

(c) multiplying any row or column of U by $e^{-i\varphi/2}$

(d) multiplying U by $e^{-i\varphi}$

Ans: (a)

Q26. The graph of the function $f(x)$ shown below is best described by



(a) The Bessel function $J_0(x)$

(b) $\cos x$

(c) $e^{-x}\cos x$

(d) $\frac{1}{x}\cos x$

Ans: (a)

Q27. In a series of five Cricket matches, one of the captains calls "Heads" every time when the toss is taken. The probability that he will win 3 times and lose 2 times is

(a) $1/8$ (b) $5/8$ (c) $3/16$ (d) $5/16$

Ans: (d)

Solution: $P=\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{5-3}\frac{5!}{3!(5-3)!}=\frac{1}{8}\times\left(\frac{1}{2}\right)^2\cdot\frac{5!}{3!(5-3)!}$

$$=\frac{1}{32}\cdot\frac{5\times4\times3!}{3!\times2!}=\frac{20}{32}=\frac{5}{8\times2}=\frac{5}{16}$$

The probability of getting exactly k successes in n trials is given by probability mass function $=\frac{n!}{k!(n-k)!}p^k\cdot(1-p)^{n-k}$, $k=$ successes, $n=$ trials.

Q28. The Taylor expansion of the function $\ln(\cosh x)$, where x is real, about the point $x=0$ starts with the following terms:

(a) $-\frac{1}{2}x^2+\frac{1}{12}x^4+....$ (b) $\frac{1}{2}x^2-\frac{1}{12}x^4+....$

(c) $-\frac{1}{2}x^2+\frac{1}{6}x^4+....$ (d) $\frac{1}{2}x^2+\frac{1}{6}x^4+....$

Ans: (b)

Solution: $\cosh x=\frac{e^x+e^{-x}}{2}$.Tailor's series expansion of $f(x)$ about $x=a$

$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + + \frac{f'''}{3!}(x-a)^3 + \ldots$. Here $a = 0$.

$$f(x) = \log\left[\frac{e^x + e^{-x}}{2}\right]\Bigg|_{x=0} = 0,\ f'(x)\big|_{x=0} = \frac{1}{\frac{e^x + e^{-x}}{2}} \cdot \frac{e^x - e^{-x}}{2} = \frac{e^x - e^x}{e^x + e^{-x}} = \tanh x = 0$$

$$f''(x) = \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} = \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2} = 1 - \tanh^2 x$$

At $x = 0$, $f''(x) = 1, f'''(x) = -2$

$$\Rightarrow f(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots\ldots.$$

Q29. The value of the integral $\int_C \frac{z^3 dz}{z^2 - 5z + 6}$, where C is a closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Ans: (a)

Solution: $z^2 - 5z + 6 = 0 \Rightarrow z^2 - 2z - 3z + 6 = 0 \Rightarrow z(z-2) - 3(z-2) = 0 \Rightarrow z = 3, 2$

$2|z| = 5 \Rightarrow |z| = 2.5$, only 2 will be inside.

$$\text{Residue} = (z-2)\frac{z^3}{(z-3)(z-2)}\Bigg|_{z=2} = \frac{8}{2-3} = -8 \Rightarrow \int_c \frac{z^3 dz}{z^2 - 5z + 6} = 2\pi i(-8) = -16\pi i$$

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Q30. Given that $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2 + 2tx}$

the value of $H_4(0)$ is

(a) 12 (b) 6 (c) 24 (d) – 6

Ans: (a)

Solution: $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx} \Rightarrow \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!} = e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!}$

$\Rightarrow \frac{H_4(0)}{4!}t^4 = \frac{t^4}{2!} \Rightarrow H_4(0) = \frac{4!}{2!} = 12.$

Q31. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120^o with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60^o if

(a) $b = \sqrt{3}a/2$ (b) $b = 2a/\sqrt{3}$ (c) $b = a/2$ (d) $b = a$

Ans: (c)

Solution: $\vec{u} = a\hat{i} + b\hat{n}$, $\vec{v} = a\hat{n} + b\hat{i}$

$\Rightarrow \vec{u}\cdot\vec{v} = (a\hat{i} + b\hat{n})\cdot(a\hat{n} + b\hat{i}) \Rightarrow |\vec{u}||\vec{v}|\cos 60 = a^2\hat{i}\cdot\hat{n} + ab + ba + b^2\hat{n}.\hat{i}$

$\left(\sqrt{a^2 + b^2 + 2ab\cos 120}\right)^2 \cdot \cos 60 = a^2\cos 120 + 2ab + b^2\cos 120$

$\left(a^2 + b^2 - 2ab\times\frac{1}{2}\right)\cdot\cos 60 = -\frac{1}{2}(a^2 + b^2) + 2ab = \frac{1}{2}(a^2 + b^2) - \frac{ab}{2} = -\frac{1}{2}(a^2 + b^2) + 2ab$

$\Rightarrow a^2 + b^2 = \frac{5ab}{2} \Rightarrow b = \frac{a}{2}.$

Q32. With $z = x + iy$, which of the following functions $f(x,y)$ is NOT a (complex) analytic function of z ?

(a) $f(x,y) = (x + iy - 8)^3(4 + x^2 - y^2 + 2ixy)^7$

(b) $f(x,y) = (x + iy)^7(1 - x - iy)^3$

(c) $f(x,y) = (x^2 - y^2 + 2ixy - 3)^5$

(d) $f(x,y) = (1 - x + iy)^4(2 + x + iy)^6$

Ans: (d)

Solution: $f(x,y) = (1 - x + iy)^4(2 + x + iy)^6$

$= \{1 - (x - iy)\}^4(2 + x + iy)^6$

Due to present of $\bar{z} = (x - iy)$

Q33. The solution of the partial differential equation

$$\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=0$$

satisfying the boundary conditions $u(0,t)=0=u(L,t)$ and initial conditions $u(x,0)=\sin(\pi x/L)$ and $\frac{\partial}{\partial t}u(x,t)\Big|_{t=0}=\sin(2\pi x/L)$ is

(a) $\sin(\pi x/L)\cos(\pi t/L)+\frac{L}{2\pi}\sin(2\pi x/L)\cos(2\pi t/L)$

(b) $2\sin(\pi x/L)\cos(\pi t/L)-\sin(\pi x/L)\cos(2\pi t/L)$

(c) $\sin(\pi x/L)\cos(2\pi t/L)+\frac{L}{\pi}\sin(2\pi x/L)\sin(\pi t/L)$

(d) $\sin(\pi x/L)\cos(\pi t/L)+\frac{L}{2\pi}\sin(2\pi x/L)\sin(2\pi t/L)$

Ans: (d)

Solution: $\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0$, $u(x,0)=\sin\frac{\pi x}{L}$ and $\frac{\partial u}{\partial t}=\sin\frac{2\pi x}{L}$

This is a wave equation

So solution is given by $u(x,t)=\sum_n\left(A_n\cos\frac{an\pi t}{L}+B_n\sin\frac{an\pi t}{L}\right)$

with $A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,\ B_n=\frac{2}{an\pi}\int_0^L g(n)\sin\frac{n\pi x}{L}dx$

Comparing $a^2\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$, We have $a=1$ and $f(x)=\sin\frac{\pi x}{L}$, $g(n)=\sin\frac{2\pi x}{L}$,

$$An=\frac{2}{L}\int_0^L\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L\sin^2\frac{\pi x}{L}dx=\frac{2}{L}\int_0^L\left(\frac{1-\cos\frac{2\pi x}{L}}{2}\right)dx=\frac{2}{L}\cdot\frac{L}{2}=1(\text{let } n=1)$$

Putting $n=2$

$$Bn=\frac{2}{an\pi}\int_0^L \sin\frac{2\pi x}{L}\cdot\sin\frac{n\pi x}{L}dx=\frac{2}{2\pi}\int_0^L \sin^2\frac{2\pi x}{L}dx=\frac{2}{2\pi}\int_0^L\left(\frac{1-\cos\frac{4\pi x}{L}}{2}\right)dx=\frac{2}{2\pi}\cdot\frac{L}{2}=\frac{L}{2\pi}$$

Q34. The solution of the differential equation

$$\frac{dx}{dt}=x^2$$

with the initial condition $x(0)=1$ will blow up as t tends to

(a) 1 (b) 2 (c) ½ (d) ∞

Ans: (a)

Solution: $\frac{dx}{dt}=x^2 \Rightarrow \int\frac{dx}{x^2}=\int dt \Rightarrow \frac{x^{-2+1}}{-2+1}=t+C \Rightarrow \frac{-1}{x}=t+C$

$\Rightarrow x(0)=1 \Rightarrow \frac{-1}{1}=0+C \Rightarrow C=-1 \Rightarrow \frac{-1}{x}=t-1 \Rightarrow x=\frac{1}{1-t}$ as $t\to 1$ x blows up

Q35. The inverse Laplace transforms of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2+1-e^{-t}$

(c) $t-1+e^{-t}$ (d) $\frac{1}{2}t^2\left(1-e^{-t}\right)$

Ans: (c)

Solution: $f(s)=\frac{1}{s+1} \Rightarrow f(t)=e^{-t} \Rightarrow L^{-1}\left[\frac{1}{s(s+1)}\right]=\int_0^t e^{-t}dt=\left(-e^{-t}\right)_0^t=\left(-e^{-t}+1\right)$

$\Rightarrow L^{-1}\left[\frac{1}{s^2(s+1)}\right]=\int_0^t\left(-e^{-t}+1\right)dt=\left(e^{-t}+t\right)_0^t=e^{-t}+t-1.$

Q36. The approximation $\cos\theta\approx 1$ is valid up to 3 decimal places as long as $|\theta|$ is less than:

(take $180^\circ/\pi\approx 57.29^\circ$)

(a) 1.28° (b) 1.81° (c) 3.28° (d) 4.01°

Ans: (b)

Solution: $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^2}{4!}-.......\approx 1-\frac{\theta^2}{2!}$

$\cos\theta \approx 1$ when $\theta = 1.81^{\circ} \approx \frac{\pi}{100} = .0314$

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Q37. If $\vec{A} = \hat{i}yz + \hat{j}xz + \hat{k}xy$, then the integral $\oint_C \vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x=0, x=a$ and $y=0,\ y=b$) is

(a) $\frac{1}{2}\left(a^3+b^3\right)$ (b) $\pi\left(ab^2+a^2b\right)$ (c) $\pi\left(a^3+b^3\right)$ (d) 0

Ans: (d)

$\oint_C \vec{A}\cdot d\vec{l} = \int_S \left(\vec{\nabla}\times\vec{A}\right) d\vec{a} = 0$ since $\vec{\nabla}\times\vec{A} = 0$.

Q38. If A, B and C are non-zero Hermitian operators, which of the following relations must be false?

(a) $[A,\ B] = C$ (b) $AB + BA = C$ (c) $ABA = C$ (d) $A + B = C$

Ans: (a)

Solution: $[A,\quad B] = C \Rightarrow AB - BA = C \quad \Rightarrow (AB-BA)^{\dagger} = C^{\dagger}$

$((AB)^{\dagger} - (BA)^{\dagger}) = C^{\dagger} \quad \Rightarrow (B^{\dagger}A^{\dagger}) - (A^{\dagger}B^{\dagger}) = C^{\dagger}$

Hence A,B and C are hermitian then

$BA - AB = C \neq [A,\ B] = C$

Q39. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?

(a) x^2y (b) $x^2 - y^2$ (c) $x^3 - 3xy^2$ (d) $3x^2y - y - y^3$

Ans: (a)

Solution: Let $x^2 y$ be real part of a complex function. Use Milne Thomson's method to write analytic complex function. The real part of that function should be (1) but that is not the case. So this cannot be real part of an analytic function. Also,

$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, Real part option (2)

$z^3 = (x+iy)^3 = x^3 - iy^3 + 3ixy(x+iy)$

$= x^3 - iy^3 + 3ix^2y - 3xy^2$, Real part option (3)

Q40. The expression

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)\frac{1}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}$$

is proportional to

(a) $\delta(x_1+x_2+x_3+x_4)$

(b) $\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$

(c) $\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{-3/2}$

(d) $\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{-2}$

Ans: (b)

Solution: $\left[\frac{\partial}{\partial x_1}\left(\frac{1}{x_1^2+x_2^2+x_3^2+x_4^2}\right)\right] = \frac{-2x_1}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^2}$

$$\frac{\partial^2}{\partial x_1^2} = -2\left[\frac{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^2\cdot 1 - 2\cdot 2x_1\cdot x_1\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^4}\right]$$

$$= -2\left[\frac{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^2 - 4x_1^2}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^3}\right] = \frac{8x_1^2 - 2\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^3}$$

Now similarly solving all and add up then we get

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}+\frac{1}{x_4^2}\right)$$

$$= \frac{8\left(x_1^2+x_2^2+x_3^2+x_4^2\right) - 8\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^3} = 0$$

also if all x_1, x_2, x_3, x_4 becomes zero it should be infinity.

So $\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)\frac{1}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}=\delta(x_1)\cdot\delta(x_2)\cdot\delta(x_3)\cdot\delta(x_4)$

Q41. Given that the integral $\int_0^\infty \frac{dx}{y^2+x^2}=\frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{\left(y^2+x^2\right)^2}$ is

(a) $\frac{\pi}{y^3}$ (b) $\frac{\pi}{4y^3}$ (c) $\frac{\pi}{8y^3}$ (d) $\frac{\pi}{2y^3}$

Ans: (b)

Solution: $\int_0^\infty \frac{dx}{\left(y^2+x^2\right)^2}=\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\left(y^2+x^2\right)^2}$, pole is of 2nd order at $x=iy$, residue $=1/\left(4iy^3\right)$

$Integral=1/2*2\pi i*1/(4iy^3)=\pi/(4y^3)$

Q42. The Fourier transform of the derivative of the Dirac δ- function, namely $\delta'(x)$, is proportional to

(a) 0 (b) 1 (c) $\sin k$ (d) ik

Ans: (d)

Solution: Fourier transform of $\delta'(x)$

$$H(K)=\int_{-\infty}^{\infty}\delta'(x)e^{ikx}dx=ike^{(k\cdot 0)}=ik$$

Q43. Consider an $n\times n(n>1)$ matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij}=ij$). The matrix A

(a) has one degenerate eigevalue with degeneracy $(n-1)$

(b) has two degenerate eigenvalues with degeneracies 2 and $(n-2)$

(c) has one degenerate eigenvalue with degeneracy n

(d) does not have any degenerate eigenvalue

Ans: (a)

Solution: If matrix is 2×2 let $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda)-4=0 \Rightarrow \lambda = 0, 5$$

If If matrix is 3×3 let $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]+2[18-2(9-\lambda)]+3[12-3(4-\lambda)]$$

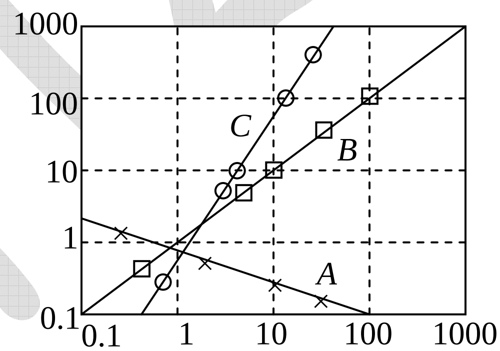
$$(1-\lambda)[\lambda^2-13\lambda+36-36]+2[18-18+2\lambda]+3[12-12+3\lambda]=0$$

$$\lambda^2-13\lambda-\lambda^3+13\lambda^2+13\lambda=0 \Rightarrow \lambda^3-14\lambda^2=0 \Rightarrow \lambda=0,0,\lambda=14$$

i.e. has one degenerate eigenvalue with degeneracy 2.

Thus one can generalized that for n dimensional matrix has one degenerate eigevalue with degeneracy $(n-1)$.

Q44. Three sets of data A, B and C from an experiment, represented by $\times$, and O, are plotted on a log-log scale. Each of these are fitted with straight lines as shown in the figure.



The functional dependence $y(x)$ for the sets A, B and C are respectively

(a) $\sqrt{x}$, x and x^2 (b) $-\frac{x}{2}$, x and $2x$ (c) $\frac{1}{x^2}$, x and x^2 (d) $\frac{1}{\sqrt{x}}$, x and x^2

Ans: (d)

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Q45. Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when t is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Ans: (b)

Solution: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0 \Rightarrow m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1, 1$

$\Rightarrow x=(c_1+c_2t)e^{-t}$ since $x(0)=0 \Rightarrow 0=c_1 \Rightarrow x=c_2te^{-t}0$

$\Rightarrow \dot{x}=c_2\left[-te^{-t}+e^{-t}\right]$

Since $\dot{x}(0)=1 \Rightarrow 1=c_2 \Rightarrow x=te^{-t}$

For maxima or minima $\dot{x}=0 \Rightarrow \dot{x}=-te^{-t}+e^{-t}=0 \Rightarrow \dot{x}=e^{-t}(1-t)$

$\Rightarrow e^{-t}=0, 1-t=0 \Rightarrow t=\infty,\ t=1$

$\ddot{x}=e^{-t}(-1)+(1-t)e^{-t}(-1)=-e^{-t}+(t-1)e^{-t}$

$\Rightarrow \ddot{x}(1)=-e^{-1}+0e^{-t}<0$

Q46. Consider the matrix

$$M=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$

The eigenvalues of M are

(a) $-5, -2, 7$ (b) $-7, 0, 7$ (c) $-4i, 2i, 2i$ (d) $2, 3, 6$

Ans: (b)

Solution: $M=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$, $M^+=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$

$M^{+} = M$

Matrix is Hermitian so roots are real and trace = 0.

$\lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0 \Rightarrow \lambda = -7, 0, 7$

Q47. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

$$\oint_C \frac{dz}{\sin^2 z}$$

is

(a) ∞ (b) $2\pi i$ (c) 0 (d) πi

Ans: (c)

Solution: $f(z) = \frac{1}{\sin^2 z}$ $\left(|z| = \frac{1}{2}\right)$

$$\sin z = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} .... \Rightarrow \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} ....\right)^2}$$

$$\Rightarrow \frac{1}{\sin^2 z} = \frac{1}{z^2}\left[1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} ....\right]^{-2} \Rightarrow \oint_C \frac{dz}{\sin^2 z} = 0$$

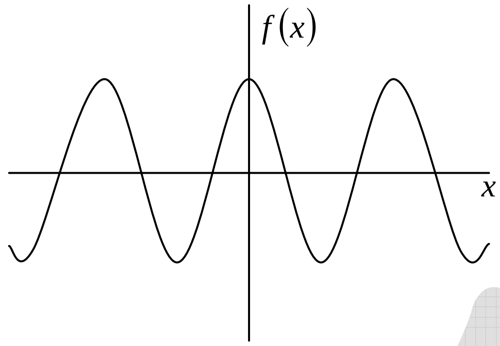
Q48. Given $\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right)^{-1/2}$, for $|t| < 1$, the value of $P_5(-1)$ is

(a) 0.26 (b) 1 (c) 0.5 (d) -1

Ans: (d)

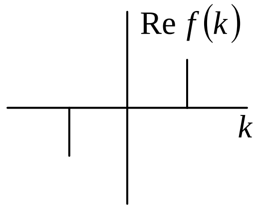
$P_n(-1) = -1$ if n is odd $\Rightarrow P_5(-1) = -1$

Q49. The graph of a real periodic function $f(x)$ for the range $[-\infty, \infty]$ is shown below

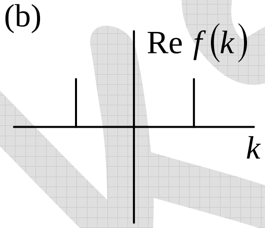


Which of the following graphs represents the real part of its Fourier transform?

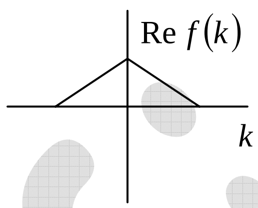
(a)



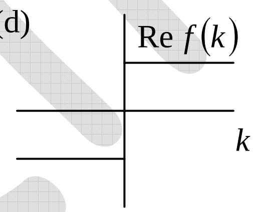
(b)



(c)



(d)



Ans: (b)

Solution: This is cosine function

$$f(x) = A\cos x \Rightarrow F(k) = \frac{A}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]$$

<u>NET/JRF (DEC-2014)</u>

Q50. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r = |\vec{r}|$. Then

(a) $\vec{\nabla}\cdot\vec{r} = 0$ and $\vec{\nabla}\times\vec{r} = \vec{r}/r$

(b) $\vec{\nabla}\cdot\vec{r} = 0$ and $\nabla^2 r = 0$

(c) $\vec{\nabla}\cdot\vec{r} = 3$ and $\nabla^2\vec{r} = \vec{r}/r^2$

(d) $\vec{\nabla}\cdot\vec{r} = 3$ and $\vec{\nabla}\times\vec{r} = 0$

Ans: (d)

Solution: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\cdot\vec{r} = = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \hat{x}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)+\hat{y}\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)+\hat{z}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right) = 0$$

Q51. The column vector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ if

(a) $b = 0$ or $a = 0$ (b) $b = a$ or $b = -2a$

(c) $b = 2a$ or $b = -a$ (d) $b = a/2$ or $b = -a/2$

Ans: (b)

Solution: *Let* $b = a$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

Let $b = -2a$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix} = \begin{pmatrix} a \\ -2a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix} = \begin{pmatrix} -a \\ 2a \\ -a \end{pmatrix} = -1\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}$$

For other combination above relation is not possible.

Q52. The principal value of the integral $\int_{-\infty}^{\infty}\frac{\sin(2x)}{x^3}dx$ is

(a) -2π (b) $-\pi$ (c) π (d) 2π

Ans: (a)

Solution: Let $f(z)=\frac{e^{i2z}}{z^3}$

$\lim_{z\to 0}(z-0)^3 f(z)=\lim_{z\to 0}(z-0)^3\frac{e^{i2z}}{z^3}=1(\text{finite and}\neq 0)\Rightarrow z=0$ is pole of order 3.

Residue $R=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[(z-0)^3\frac{e^{iz}}{z^3}\right]=-2$

$\Rightarrow\int_{-\infty}^{\infty}f(x)dx=\pi i\Sigma R=\pi i(-2)=-2\pi i\Rightarrow \text{Im}.Part=-2\pi\Rightarrow\int_{-\infty}^{\infty}f(x)dx=-2\pi$

Q53. The Laurent series expansion of the function $f(z)=e^z+e^{1/z}$ about $z=0$ is given by

(a) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ for all $|z|<\infty$

(b) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ only if $0<|z|<1$

(c) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ for all $0<|z|<\infty$

(d) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ only if $|z|<1$

Ans: (c)

Solution: $e^z=\left(1+z+\frac{z^2}{2!}+....\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ and $e^{1/z}=1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+....=\sum_{n=0}^{\infty}\frac{1}{z^n n!}$

$\Rightarrow f(z)=e^z+e^{1/z}=\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n\ !}$ for all $0<|z|<\infty$

Q54. Two independent random variables m and n, which can take the integer values $0, 1, 2, ..., \infty$, follow the Poisson distribution, with distinct mean values μ and ν respectively. Then

(a) the probability distribution of the random variable $l=m+n$ is a binomial distribution.

(b) the probability distribution of the random variable $r=m-n$ is also a Poisson distribution.

(c) the variance of the random variable $l=m+n$ is equal to $\mu+\nu$

(d) the mean value of the random variable $r=m-n$ is equal to 0.

Ans: (c)

Solution: $\sigma_l^2=\sigma_m^2+\sigma_n^2=\mu+\nu$

Q55. Consider the function $f(z)=\frac{1}{z}\ln(1-z)$ of a complex variable $z=re^{i\theta}(r\geq 0,\ -\infty<\theta<\infty)$. The singularities of $f(z)$ are as follows:

(a) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ only for $0\leq\theta<2\pi$

(b) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ other than $0\leq\theta<2\pi$

(c) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ

(d) branch points at $z=0,\ z=1$ and $z=\infty$.

Ans: None of the above is correct

Solution: For $f(z)=\frac{1}{z}\ln(1-z)=\frac{1}{z}\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-.....\right)=-1-\frac{z}{2}-\frac{z^2}{3}-.....$

There is no principal part and when $z\to 0$, $f(z)=-1$. So there is removable singularity at $z=0$. Also $z=1$ and $z=\infty$ is Branch point.

Q56. The function $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$ satisfies the differential equation

(a) $x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2+1)f=0$

(b) $x^2\frac{d^2f}{dx^2}+2x\frac{df}{dx}+(x^2-1)f=0$

(c) $x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2-1)f=0$

(d) $x^2\frac{d^2f}{dx^2}-x\frac{df}{dx}+(x^2-1)f=0$

Ans: (c)

Solution: $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$ is generating function (Bessel Function of first kind) which satisfies the differential equation $x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2-n^2)f=0$, put $n=1$.

Q57. Let α and β be complex numbers. Which of the following sets of matrices forms a group under matrix multiplication?

(a) $\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix}$

(b) $\begin{pmatrix}1 & \alpha\\ \beta & 1\end{pmatrix}$, where $\alpha\beta\neq 1$

(c) $\begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix}$, where $\alpha\beta^*$ is real

(d) $\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}$, where $|\alpha|^2+|\beta|^2=1$

Ans: (d)

Solution: $\because \begin{vmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{vmatrix} = |\alpha|^2 + |\beta|^2 = 1$

Q58. The expression $\sum_{i,j,k=1}^{3} \in_{ijk} \{x_i, \{p_j, L_k\}\}$ (where $\in_{ijk}$ is the Levi-Civita symbol, $\vec{x}, \vec{p}, \vec{L}$ are the position, momentum and angular momentum respectively, and $\{A, B\}$ represents the Poisson Bracket of A and B) simplifies to

(a) 0 (b) 6 (c) $\vec{x}, (\vec{p} \times \vec{L})$ (d) $\vec{x} \times \vec{p}$

Ans: (b)

NET/JRF (JUNE-2015)

Q59. The value of integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\pi}{2}$ (c) $\sqrt{2}\pi$ (d) 2π

Ans. (a)

Solution: $\int_{-\infty}^{\infty} \frac{dz}{1+z^4} \quad \because |z| = R$

Now pole $\Rightarrow \quad z = e^{(2n+1)\frac{\pi}{4}}$

$n = 0, \quad \Rightarrow z_0 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n = 2 \Rightarrow z_2 = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$n = 1 \Rightarrow z_1 = e^{\frac{i3\pi}{4}} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n = 3 \Rightarrow z_3 = +\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

only z_0 and z_1 lies in contour

i.e., residue at $z = e^{\frac{i\pi}{4}} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

residue at $z = e^{\frac{i3\pi}{4}} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

now $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \varepsilon \operatorname{Re} s = \frac{\pi}{\sqrt{2}}$

Q60. Consider the differential equation $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$. If $x=0$ at $t=0$ and $x=1$ at $t=1$, the value of x at $t=2$ is

(a) e^2+1 (b) e^2+e (c) $e+2$ (d) $2e$

Ans. (b)

Solution: $D^2-3D+2=0$

$(D-1)(D-2)=0 \Rightarrow D=1,2 \qquad \Rightarrow x=c_1e^{2t}+c_2e^t$

using boundary condition $x=0, t=0$

$c_1=c_2$

again using boundary condition $x=1, t=1$

$$c_2=\frac{1}{e-e^2}, c_1=\frac{1}{e^2-e} \Rightarrow x=\frac{e^{2t}}{e^2-e}+\frac{1}{e-e^2}e^t$$

again using $t=2$ then $x=e^2+e$

Q61. The Laplace transform of $6t^3+3\sin 4t$ is

(a) $\frac{36}{s^4}+\frac{12}{s^2+16}$ (b) $\frac{36}{s^4}+\frac{12}{s^2-16}$

(c) $\frac{18}{s^4}+\frac{12}{s^2-16}$ (d) $\frac{36}{s^3}+\frac{12}{s^2+16}$

Ans. (a)

Solution: $L\left[6t^3+3\sin 4t\right] \qquad \because L\left[t^n\right]=\frac{\lceil n+1}{S^{n+1}}$

$$\because L\sin at=\frac{a}{s^2+a^2}$$

$$\Rightarrow \qquad \frac{6\times\lceil 4}{s^4}+\frac{3\times 4}{s^2+16} \Rightarrow \frac{36}{s^4}+\frac{12}{s^2+16}$$

Q62. Let $f(x,t)$ be a solution of the wave equation $\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 f}{\partial x^2}$ in 1-dimension. If at $t=0, f(x,0)=e^{-x^2}$ and $\frac{\partial f}{\partial t}(x,0)=0$ for all x, then $f(x,t)$ for all future times $t>0$ is described by

(a) $e^{-\left(x^2-v^2t^2\right)}$

(b) $e^{-(x-vt)^2}$

(c) $\frac{1}{4}e^{-(x-vt)^2}+\frac{3}{4}e^{-(x+vt)^2}$

(d) $\frac{1}{2}\left[e^{-(x-vt)^2}+e^{-(x+vt)^2}\right]$

Ans. (d)

Solution: For $\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 f}{\partial x^2}$

$\frac{\partial f}{\partial t}(x,0)=0$ and $f(x,0)=e^{-x^2}$

$f=\frac{1}{2}\left[f(x+vt)+f(x-vt)\right]$

therefore, solution is $f=\frac{1}{2}\left(e^{-(x-vt)}\right)^2+e^{-(x+vt)^2}$

NET/JRF (DEC-2015)

Q63. In the scattering of some elementary particles, the scattering cross-section σ is found to depend on the total energy E and the fundamental constants h (Planck's constant) and c (the speed of light in vacuum). Using dimensional analysis, the dependence of σ on these quantities is given by

(a) $\sqrt{\frac{hc}{E}}$

(b) $\frac{hc}{E^{3/2}}$

(c) $\left(\frac{hc}{E}\right)^2$

(d) $\frac{hc}{E}$

Ans.: (c)

Solution: The dimension of σ is dimension of "Area"

$h = Joul - \sec$

$c = m/\sec$

$E = Joul$

$\left(\frac{hc}{E}\right)^2 = m^2$ dimension of area

Q64. If $y = \frac{1}{\tanh(x)}$, then x is

(a) $\ln\left(\frac{y+1}{y-1}\right)$ (b) $\ln\left(\frac{y-1}{y+1}\right)$

(c) $\ln\sqrt{\frac{y-1}{y+1}}$ (d) $\ln\sqrt{\frac{y+1}{y-1}}$

Ans.: (d)

Solution: $y = \frac{1}{\tanh x}$

$$y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

$$ye^{2x} - y = e^{2x} + 1 \Rightarrow ye^{2x} - e^{2x} = 1 + y \Rightarrow e^{2x}(y-1) = 1 + y$$

$$2x = \ln\left(\frac{y+1}{y-1}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) = \ln\left(\frac{y+1}{y-1}\right)^{\frac{1}{2}}$$

Q65. The function $\frac{z}{\sin \pi z^2}$ of a complex variable z has

(a) a simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ for $n = 1, 2, 3...$

(b) a simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ and $\pm i\sqrt{n}$ for $n = 1, 2, 3...$

(c) poles of order 2 at $\pm\sqrt{n}$, $n = 0, 1, 2, 3...$

(d) poles of order 2 at $\pm n$, $n = 0, 1, 2, 3...$

Ans. : (b)

Solution: $f(z)=\dfrac{z}{\sin \pi z^2}=\dfrac{z}{\pi z^2 \dfrac{\sin \pi z^2}{\pi z^2}}$

at $z=0$ these is a simple pole since $\lim\limits_{z\to 0}\dfrac{\sin \pi z^2}{\pi z^2}=1$

Also, $\sin \pi z^2 = \sin n\pi$ $\lim\limits_{z\to\sqrt{n}}\left(z-\sqrt{n}\right)^2 . \dfrac{z}{\sin \pi z^2}$

$\pi z^2 = \pm n\pi \qquad z=\pm\sqrt{n}, \pm i\sqrt{n}$

exists so its pole of order 2.

Q66. The Fourier transform of $f(x)$ is $\tilde{f}(k)=\int_{-\infty}^{+\infty} dx e^{ikx} f(x)$.

If $f(x)=\alpha\delta(x)+\beta\delta'(x)+\gamma\delta''(x)$, where $\delta(x)$ is the Dirac delta-function (and prime denotes derivative), what is $\tilde{f}(k)$?

(a) $\alpha+i\beta k+i\gamma k^2$
(b) $\alpha+\beta k-\gamma k^2$
(c) $\alpha-i\beta k-\gamma k^2$
(d) $i\alpha+\beta k-i\gamma k^2$

Ans.: (c)

Solution: $\tilde{f}(k)=\int\limits_{-\infty}^{\infty} dx\, e^{ikx}\left(\alpha\delta(x)+\beta\delta'(x)+\gamma\delta''(x)\right)$

$$\int\limits_{-\infty}^{\infty}\alpha\delta(x)e^{ikx}dx=\alpha$$

$$\int\limits_{-\infty}^{\infty}\beta\delta'(x)e^{ikx}dx=\beta\left[e^{ikx}\delta(x)\Big|_{-\infty}^{\infty}-\int\limits_{-\infty}^{\infty}ike^{ikx}\delta(x)dx\right]=-i\beta k$$

$$\int\limits_{-\infty}^{\infty}\gamma\delta''(x)e^{ikx}dx=-\gamma k^2$$

Q67. The solution of the differential equation $\frac{dx}{dt} = 2\sqrt{1-x^2}$, with initial condition $x = 0$ at $t = 0$ is

(a) $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ \sinh 2t, & t \ge \frac{\pi}{4} \end{cases}$

(b) $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{2} \\ 1, & t \ge \frac{\pi}{2} \end{cases}$

(c) $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ 1, & t \ge \frac{\pi}{4} \end{cases}$

(d) $x = 1 - \cos 2t, \quad t \ge 0$

Ans.: (c)

Solution: $\frac{dx}{dt} = 2\sqrt{1-x^2}$

$\frac{dx}{\sqrt{1-x^2}} = 2dt \qquad \sin^{-1} x = 2t + c$

$x = 0,\ t = 0 \qquad \text{so, } c = 0$

$x = \sin 2t$

x should not be greater than 1 at $x = 1$

$1 = \sin 2t \qquad \sin\frac{\pi}{2} = \sin 2t,\ t = \frac{\pi}{4}$

so, $\quad x = \sin 2t \qquad 0 \le t < \frac{\pi}{4}$

$\qquad = 1 \qquad t \ge \frac{\pi}{4}$

Q68. The Hermite polynomial $H_n(x)$ satisfies the differential equation

$$\frac{d^2H_n}{dx^2}-2x\frac{dH_n}{dx}+2nH_n(x)=0$$

The corresponding generating function

$G(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)t^n$ satisfies the equation

(a) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$

(b) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}-2t^2\frac{\partial G}{\partial t}=0$

(c) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial G}{\partial t}=0$

(d) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial^2 G}{\partial x\partial t}=0$

Ans.: (a)

Solution: $G=\frac{1}{n!}H_n(x)t^n$

$$G'=\frac{1}{n!}H_n'(x)t^n$$

$$G''=\frac{1}{n!}H_n''(x)t^n$$

$$\frac{\partial G}{\partial t}=\frac{1}{n!}H_n(x)nt^{n-1}$$

lets check the options one by one

$$\frac{\partial G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$$

$$\frac{1}{n!}H_n''(x)t^n-2x\frac{1}{n!}H_n'(x)t^n+2t\frac{1}{n!}H_n(x)n$$

$H_n''(x)-2xH_n'(x)+2xH_n(x)=0$, which is Hermite Differential Equation.

Q69. The value of the integral $\int_0^8 \frac{1}{x^2+5}dx$, valuated using Simpson's $\frac{1}{3}$ rule with $h=2$ is

(a) 0.565 (b) 0.620 (c) 0.698 (d) 0.736

Ans.: (a)

Solution: $I=\frac{2}{3}\left[y_0+4\left(y_1+y_2\right)+2y_2+y_4\right]$

$=\frac{2}{3}\left[\frac{1}{5}+4\left(\frac{1}{9}+\frac{1}{4}\right)+2\times\frac{1}{21}+\frac{1}{69}\right]$

$=\frac{2}{3}\left[\frac{1}{5}+0.5734+0.09523+0.0145\right]$

$=\frac{2}{3}\left[0.2+0.5734+0.09523+0.0145\right]$

$=\frac{2}{3}\times 0.8831=0.5887$

x	$y=\frac{1}{x^2+5}$
0	$\frac{1}{5}$
2	$\frac{1}{9}$
4	$\frac{1}{21}$
6	$\frac{1}{31}$
8	$\frac{1}{69}$

Q70. Consider a random walker on a square lattice. At each step the walker moves to a nearest neighbour site with equal probability for each of the four sites. The walker starts at the origin and takes 3 steps. The probability that during this walk no site is visited more than one is

(a) $12/27$ (b) $27/64$ (c) $3/8$ (d) $9/16$

Ans.: (d)

Solution: Total number of ways $=4\times4\times4$

Number of preferred outcome $=4\times3\times3$

($\because$ Any four option in step-1 and only 3 option in step $2\&3$ because he can not go to previous position)

probability $=\frac{4\times3\times3}{4\times4\times4}=\frac{9}{16}$